

A KUROSH TYPE THEOREM FOR TYPE II_1 FACTORS

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ABSTRACT. We prove a Kurosh type theorem for free-product type II_1 factors. In particular, if $M = L\mathbb{F}_2 \bar{\otimes} \mathcal{R}$, then the free-product type II_1 factors $M * \dots * M$ are all prime and pairwise non-isomorphic. We also study the case of crossed product type II_1 factors. This paper is a continuation of our previous papers [Oz2][OP], where the structure of (tensor products of) word hyperbolic group type II_1 factors was studied.

1. INTRODUCTION

The classification of type II_1 factors (of discrete groups) was initiated by Murray and von Neumann [MvN] who distinguished the hyperfinite type II_1 factor \mathcal{R} from the group factor $L\mathbb{F}_r$ of the free group \mathbb{F}_r on $r \geq 2$ generators. Thirty years later, Connes [Co2] proved uniqueness of the injective type II_1 factor. Thus, the group factor $L\Gamma$ of an ICC amenable group Γ is isomorphic to the hyperfinite type II_1 factor \mathcal{R} . On the other hand, *the* isomorphism problem of free group factors remains open. To solve this problem, Voiculescu invented free probability theory, which led to a number of deep results on the structure of free group factors (cf. the survey paper [Vo2]). Apart from these results and results of Connes [Co1] and Cowling and Haagerup [CH], the classification of type II_1 factors has been vague by and large. Recently, however, a breakthrough came when Popa [Po2][Po4] found that unitary conjugacy results can be deduced from existence of finite-dimensional bimodules and obtained quite precise classification theorems for certain classes of type II_1 factors. On the other hand, a C^* -algebraic method [Oz2] was proved to be useful in study of type II_1 factors. These methods in combination yielded some prime factorization results in [OP].

This paper is a continuation of [Oz2] and [OP], where the structure of (tensor products of) word hyperbolic group type II_1 factors was studied. In this paper, we will study the structure of free-products and crossed products of certain type II_1 factors. A crucial ingredient of the argument is a computation of the kernels of certain morphisms on C^* -algebras. The idea of exploiting a ‘boundary’ to compute

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such kernels is due to Skandalis [Sk] and developed by Higson and Guentner [HG]. We will take advantage of this idea. We denote by \mathcal{S} the class of countable discrete groups Γ such that the left and right translation action of $\Gamma \times \Gamma$ on the Stone-Ćech remainder $\partial^\beta \Gamma = \beta \Gamma \setminus \Gamma$ is amenable (see Section 4 for details). The class \mathcal{S} was suggested by Skandalis and it contains all subgroups of word hyperbolic groups and discrete subgroups of connected simple Lie groups of rank one [HG][Sk]. The class \mathcal{S} also contains a group with an infinite amenable normal subgroup (cf. Corollary 4.5). The main result of [Oz2] was solidity of the group factor $L\Gamma$ of a group Γ in \mathcal{S} . The q -Gaussian von Neumann algebras (for certain values of q) are other examples of solid factors. Indeed, solidity for certain values of q was proved by Shlyakhtenko [Sh] while factoriality for all values of q was proved by Ricard [Ri]. The main results of [OP] were unique prime factorization and rigidity of their tensor products.

The main result of this paper is a Kurosh type theorem for a free-product of certain type II_1 factors. Although the theorem is not as precise as the original Kurosh theorem in group theory, it implies, for instance, that the iterated free-product type II_1 factors

$$L\mathbb{F}_\infty * (L\mathbb{F}_\infty \bar{\otimes} \mathcal{R})^{*n}, \quad n = 1, 2, \dots$$

are mutually non-isomorphic. This is a contrast to Dykema's theorem (Theorem 3.5 in [Dy1]) that $L\mathbb{F}_\infty * (L\mathbb{F}_\infty \bar{\otimes} L^\infty[0, 1])^{*n}$ are all isomorphic. There is an obvious similarity between these results and the isomorphism problem of free group factors. In fact, according to Dykema and Rădulescu [DR], isomorphism of free group factors would imply that $\mathcal{M}_1 * \mathcal{M}_2 = \mathcal{M}_1 * \mathcal{M}_2 * L\mathbb{F}_\infty$ for any type II_1 factors \mathcal{M}_1 and \mathcal{M}_2 . The proof of the above theorem consists of an adaptation for free-product of the method developed in [Oz2] and [OP] (which we will review in Section 2) and Popa's work [Po1] on normalizers in a free-product.

Definition 1.1. A type II_1 factor \mathcal{M} is *prime* if $\mathcal{M} \neq \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ for any type II_1 factors \mathcal{M}_1 and \mathcal{M}_2 . A (finite) von Neumann algebra \mathcal{M} is *solid* if for any diffuse von Neumann subalgebra \mathcal{A} , the relative commutant $\mathcal{A}' \cap \mathcal{M}$ is injective. A (finite) von Neumann algebra is *semisolid* if for any type II_1 von Neumann subalgebra \mathcal{Q} , the relative commutant $\mathcal{Q}' \cap \mathcal{M}$ is injective. A von Neumann algebra is *semiexact* if it contains a ultraweakly dense exact C^* -algebra.

There are obvious implications; $\text{solid} \Rightarrow \text{semisolid} \Rightarrow \text{prime}$ for a non-injective type II_1 factor. We will see none of these implications is reversible. For a technical reason, the results in this paper are valid only for semiexact von Neumann algebras. There are plenty of semiexact von Neumann algebras. A discrete group Γ is exact if and only if its reduced group C^* -algebra $C_\lambda^*\Gamma$ is an exact C^* -algebra (cf. [KW]). Thus, by definition, the group von Neumann algebra $L\Gamma$ of a discrete exact group Γ is semiexact. The above mentioned Kurosh type theorem implies that every free-product of semiexact type II_1 factors is prime. This gives an example of prime type II_1 factors which are not semisolid.

We also deal with crossed product and prove that the group-measure space von Neumann algebra $\Gamma \ltimes L^\infty[0, 1]$ of a measure-preserving action of a group Γ on the standard probability space $[0, 1]$ is semisolid provided that the group Γ is in \mathcal{S} . This generalizes Adams' theorem [Ad] that a measurable orbit equivalence relation of a non-amenable hyperbolic group is indecomposable. This also gives an example of semisolid type II₁ factors with the property (Γ). Note that a type II₁ factor with the property (Γ) cannot be solid by Proposition 7 in [Oz2].

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2. CONVENTIONS AND PRELIMINARY BACKGROUND

For a discrete group Γ , we denote by λ (resp. ρ) the left (resp. right) regular representation of Γ on $\ell_2\Gamma$. The reduced group C^* -algebra $C_\lambda^*\Gamma$ (resp. $C_\rho^*\Gamma$) is the C^* -subalgebra in $\mathbb{B}(\ell_2\Gamma)$ which is generated by $\lambda(\Gamma)$ (resp. $\rho(\Gamma)$) and the group von Neumann algebra $L\Gamma$ is the von Neumann algebra generated by $\lambda(\Gamma)$. We denote by $C_{\lambda,\rho}^*\Gamma$ the C^* -subalgebra in $\mathbb{B}(\ell_2\Gamma)$ which is generated by $C_\lambda^*\Gamma$ and $C_\rho^*\Gamma$. Given a finite von Neumann algebra \mathcal{M} , we assume that there is a distinguished faithful normal trace τ on \mathcal{M} and the trace on its von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$ is the restriction of τ on \mathcal{N} . So, we will write $L^2\mathcal{M}$ without specifying the trace τ . We denote by \widehat{a} the vector in $L^2\mathcal{M}$ corresponding to $a \in \mathcal{M}$. The Hilbert space $L^2\mathcal{M}$ is an \mathcal{M} - \mathcal{M} bimodule with $a\widehat{x}b = \widehat{axb}$ for $a, b, x \in \mathcal{M}$. The canonical conjugation $J_{\mathcal{M}}$ on $L^2\mathcal{M}$ is given by $J\widehat{a} = \widehat{a^*}$ for $a \in \mathcal{M}$. We simply denote $J_{\mathcal{M}}$ by J if there are no confusions. We will use the same notations for a C^* -algebra with a faithful trace. When dealing with C^* -algebras, the symbol \otimes means the algebraic tensor product while \otimes_{\min} means the minimal (spatial) tensor product. The spatial tensor product of von Neumann algebras is denoted by $\bar{\otimes}$ and the Hilbert space tensor product of Hilbert spaces is simply denoted by \otimes . The term ‘ucp’ is an abbreviation for ‘unital completely positive’. All representations and homomorphisms are assumed to be self-adjoint and non-degenerate. All groups denoted by Γ and Δ are assumed to be countable and discrete. All von Neumann algebras are assumed to have separable predual.

We review the method developed in [Oz2] and [OP]. It tells when von Neumann subalgebra \mathcal{Q} in a finite von Neumann algebra \mathcal{M} has the relative commutant $\mathcal{Q}' \cap \mathcal{M}$ that is not injective.

Let $\mathcal{M} \subset \mathbb{B}(L^2\mathcal{M})$ be a finite von Neumann algebra and $\mathcal{Q} \subset \mathcal{M}$ be an injective von Neumann subalgebra. Then, there exists a conditional expectation $\Psi_{\mathcal{Q}}$ from $\mathbb{B}(L^2\mathcal{M})$ onto \mathcal{Q}' which is proper, i.e.,

$$\forall x \in \mathbb{B}(L^2\mathcal{M}) \quad \Psi_{\mathcal{Q}}(x) \in \overline{\text{conv}}^w\{uxu^* : u \in \mathcal{U}(\mathcal{Q})\}.$$

It follows that $\Psi_{\mathcal{Q}}|_{\mathcal{M}'} = \text{id}_{\mathcal{M}'}$ and that $\Psi_{\mathcal{Q}}|_{\mathcal{M}}$ is a trace preserving conditional expectation from \mathcal{M} onto $\mathcal{Q}' \cap \mathcal{M}$, which coincides with the unique trace preserving conditional expectation $E_{\mathcal{Q}' \cap \mathcal{M}}$ from \mathcal{M} onto $\mathcal{Q}' \cap \mathcal{M}$. Since $\Psi_{\mathcal{Q}}|_{\mathcal{M}'} = \text{id}_{\mathcal{M}'}$, the ucp map $\Psi_{\mathcal{Q}}$ is an \mathcal{M}' -bimodule map;

$$\forall x, y \in \mathcal{M}', \forall b \in \mathbb{B}(L^2\mathcal{M}) \quad \Psi_{\mathcal{Q}}(xb y) = x\Psi_{\mathcal{Q}}(b)y.$$

In particular, we have

$$\forall a \in \mathcal{M}, \forall x \in \mathcal{M}' \quad \Psi_{\mathcal{Q}}(ax) = E_{\mathcal{Q}' \cap \mathcal{M}}(a)x.$$

Now Lemma 5 in [Oz2] can be interpreted as follows;

Lemma 2.1 ([Oz2]). *Let $\mathcal{M} \subset \mathbb{B}(L^2\mathcal{M})$ be a finite von Neumann algebra with an injective von Neumann subalgebra $\mathcal{Q} \subset \mathcal{M}$. If there exist unital ultraweakly dense C^* -subalgebras $B \subset \mathcal{M}$ and $C \subset \mathcal{M}'$ with B exact such that the ucp map*

$$\tilde{\Psi}_{\mathcal{Q}}: B \otimes C \ni \sum_{k=1}^n a_k \otimes x_k \mapsto \Psi_{\mathcal{Q}}\left(\sum_{k=1}^n a_k x_k\right) \in \mathbb{B}(\mathcal{H})$$

is continuous w.r.t. the minimal tensor norm on $B \otimes C$, then the relative commutant $\mathcal{Q}' \cap \mathcal{M}$ is injective.

As in [OP], in an actual application, non-injectivity of $\mathcal{Q}' \cap \mathcal{M}$ forces

$$\mathbb{K}(\mathcal{K}) \otimes_{\min} \mathbb{B}(L^2\mathcal{N}) \not\subset \ker \Psi_{\mathcal{Q}}$$

for some von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$ and a Hilbert space \mathcal{K} such that $L^2\mathcal{M} = \mathcal{K} \otimes L^2\mathcal{N}$ as a right \mathcal{N} module. If this is the case, we may find a finite rank projection p on \mathcal{K} with $b = \Psi_{\mathcal{Q}}(p \otimes 1_{\mathcal{N}}) \neq 0$. Since $\Psi_{\mathcal{Q}}$ is proper, b commutes with the right \mathcal{N} action, or equivalently $b \in (\mathbb{B}(\mathcal{K}) \bar{\otimes} \mathcal{N}) \cap \mathcal{Q}'$. By Proposition 1.3.2 in [Po2], we have $(\text{Tr} \otimes \tau_{\mathcal{N}})(b) \leq \text{Tr}(p) < \infty$. Thus, there is a non-zero spectral projection e of b with $(\text{Tr} \otimes \tau_{\mathcal{N}})(e) < \infty$. It follows that $\mathcal{H} = eL^2\mathcal{M}$ is a \mathcal{Q} - \mathcal{N} sub-bimodule of $L^2\mathcal{M}$ with $\dim_{\mathcal{N}} \mathcal{H}_{\mathcal{N}} < \infty$. (Strictly speaking, \mathcal{H} is a $\mathcal{Q}e\mathcal{N}e'$ bimodule where $e' = J_{\mathcal{M}}eJ_{\mathcal{M}}$.) If in addition \mathcal{M} and \mathcal{N} are factors, then we can apply the following Lemma 5 in [Po3] and Proposition 12 in [OP].

Lemma 2.2 ([OP][Po3]). *Let \mathcal{N} and \mathcal{Q} be subfactors in a type II_1 factor \mathcal{M} . If there exists a non-zero \mathcal{Q} - \mathcal{N} sub-bimodule $\mathcal{H} \subset L^2\mathcal{M}$ with $\dim_{\mathcal{N}} \mathcal{H}_{\mathcal{N}} < \infty$, then there exist projections $e \in \text{Proj}(\mathcal{N})$ and $q \in \text{Proj}(\mathcal{Q})$, a non-zero partial isometry $v \in \mathcal{M}$ and a homomorphism $\theta: q\mathcal{Q}q \rightarrow e\mathcal{N}e$ such that*

$$vv^* \in (q\mathcal{Q}q)' \cap q\mathcal{M}q, \quad v^*v \in \theta(q\mathcal{Q}q)' \cap e\mathcal{M}e \quad \text{and} \quad xv = v\theta(x) \quad \text{for } x \in q\mathcal{Q}q.$$

We need two more lemmas. The first is about normalizers in a free-product due to Popa. The following is formally stronger than stated in Theorem 6.1 and Corollary 4.3 in [Po1], but their proofs are same (along Proposition 4.1 in [Po1]).

Lemma 2.3 ([Po1]). *Let $\mathcal{N}_1 \subset \mathcal{M}_1$ and \mathcal{M}_2 be finite von Neumann algebras and let $\mathcal{M} = \mathcal{M}_1 * \mathcal{M}_2$ be their free-product. If \mathcal{N}_1 is diffuse and a unitary operator $u \in \mathcal{M}$ satisfies $u^* \mathcal{N}_1 u \subset \mathcal{M}_i$ for some i , then $i = 1$ and $u \in \mathcal{M}_1$.*

The last lemma in this section is about nuclearity of reduced free-product C^* -algebras. We recall that a ucp map $\varphi: A \rightarrow B$ is said to be *nuclear* if there exist nets of ucp maps $\beta^\lambda: A \rightarrow \mathbb{M}_{n(\lambda)}(\mathbb{C})$ and $\alpha^\lambda: \mathbb{M}_{n(\lambda)}(\mathbb{C}) \rightarrow B$ such that $\alpha^\lambda \circ \beta^\lambda \rightarrow \varphi$ in the point-norm topology. If $\varphi: A \rightarrow B$ is a nuclear ucp map and $B \subset \mathbb{B}(\mathcal{H})$, then the ucp map

$$\varphi \times \text{id}_{B'}: A \otimes B' \ni \sum a_k \otimes x_k \mapsto \sum \varphi(a_k) x_k \in \mathbb{B}(\mathcal{H})$$

is continuous w.r.t. the minimal tensor norm on $A \otimes B'$.

Lemma 2.4. *Let $B_i \subset \mathbb{B}(\mathcal{H}_i)$ be a C^* -subalgebra with a B_i -cyclic unit vector $\xi_i \in \mathcal{H}_i$. We denote by ω_i the vector state corresponding to ξ_i . If both B_i are exact, then the inclusion map of $(B_1, \omega_1) * (B_2, \omega_2)$ into $(\mathbb{B}(\mathcal{H}_1), \omega_1) * (\mathbb{B}(\mathcal{H}_2), \omega_2)$ is nuclear.*

Proof. It suffices to show that the inclusion map is approximated by ucp maps which factor through nuclear C^* -algebras. We first note that if $B \subset \mathbb{B}(\mathcal{H})$ is exact, then so is $B + \mathbb{K}(\mathcal{H})$. (This is well-known and the proof is involved. Indeed, by Kirchberg's theorem [Ki1], B is locally reflexive and the quotient $C := B/(B \cap \mathbb{K}(\mathcal{H}))$ is exact. Moreover, the short exact sequence

$$0 \rightarrow \mathbb{K}(\mathcal{H}) \rightarrow B + \mathbb{K}(\mathcal{H}) \rightarrow C \rightarrow 0$$

has ucp local splittings by the Effros-Haagerup lifting theorem [EH]. Now, the exactness of $B + \mathbb{K}(\mathcal{H})$ follows from that of $\mathbb{K}(\mathcal{H})$ and C by the 3-by-3 lemma.) By replacing B_i with $B_i + \mathbb{K}(\mathcal{H}_i)$, we may assume that $\mathbb{K}(\mathcal{H}_i) \subset B_i$. Since B_i is exact, there are a net of finite dimensional subspaces $\mathcal{K}_i^\lambda \subset \mathcal{H}_i$ with $\xi_i \in \mathcal{K}_i^\lambda$ with the corresponding compression $\beta_i^\lambda: B_i \rightarrow \mathbb{B}(\mathcal{K}_i^\lambda)$, and a net of ucp maps $\alpha_i^\lambda: \mathbb{B}(\mathcal{K}_i^\lambda) \rightarrow \mathbb{B}(\mathcal{H}_i)$ such that the net $\alpha_i^\lambda \circ \beta_i^\lambda$ converges pointwise to the inclusion $B_i \hookrightarrow \mathbb{B}(\mathcal{H}_i)$. Since the rank-one projection p_i corresponding to ξ_i is in B_i , we have $\lim_\lambda \alpha_i^\lambda(p_i) = p_i$. Thus, by perturbing α_i^λ , we may assume that $\alpha_i^\lambda(p_i) = p_i$ for all λ . It follows from Choda-Blanchard-Dykema's theorem [BD] that

$$\beta_1^\lambda * \beta_2^\lambda: (B_1, \omega_1) * (B_2, \omega_2) \rightarrow (\mathbb{B}(\mathcal{K}_1^\lambda), \omega_1) * (\mathbb{B}(\mathcal{K}_2^\lambda), \omega_2)$$

and

$$\alpha_1^\lambda * \alpha_2^\lambda: (\mathbb{B}(\mathcal{K}_1^\lambda), \omega_1) * (\mathbb{B}(\mathcal{K}_2^\lambda), \omega_2) \rightarrow (\mathbb{B}(\mathcal{H}_1), \omega_1) * (\mathbb{B}(\mathcal{H}_2), \omega_2)$$

are ucp maps such that the net $(\alpha_1^\lambda * \alpha_2^\lambda) \circ (\beta_1^\lambda * \beta_2^\lambda)$ converges pointwise to the inclusion map $(B_1, \omega_1) * (B_2, \omega_2) \hookrightarrow (\mathbb{B}(\mathcal{H}_1), \omega_1) * (\mathbb{B}(\mathcal{H}_2), \omega_2)$. Since the C^* -algebras $(\mathbb{B}(\mathcal{K}_1^\lambda), \omega_1) * (\mathbb{B}(\mathcal{K}_2^\lambda), \omega_2)$ are nuclear (see e.g., [Ki2] or [Oz4]), we are done. \square

3. FREE-PRODUCT

We recall the reduced free-product construction (in the tracial setting). Let B_i , $i \in \{1, 2\}$ be a C^* -algebra with a faithful trace τ_i . Let B_i act on the GNS-Hilbert space $\mathcal{H}_i = L^2(B_i, \tau_i)$. We denote by $\widehat{a} \in \mathcal{H}_i$ the vector associated with $a \in B_i$ and denote by $\xi_i = \widehat{1} \in \mathcal{H}_i$ the cyclic separating trace vector for B_i . The canonical conjugation J_i on \mathcal{H}_i is given by $J_i \widehat{a} = \widehat{a}^*$. Let $B_i^0 = \ker \tau_i \subset B_i$ and let $\mathcal{H}_i^0 = \mathcal{H}_i \ominus \mathbb{C}\xi_i$ be the closure of B_i^0 in \mathcal{H}_i . Then the free-product Hilbert space is

$$\mathcal{H} = \mathbb{C}\xi \oplus \bigoplus_{n \geq 1} \bigoplus_{\substack{i_1, \dots, i_n \in \{1, 2\}, \\ i_1 \neq i_2, \dots, i_{n-1} \neq i_n}} \mathcal{H}_{i_1}^0 \otimes \mathcal{H}_{i_2}^0 \otimes \cdots \otimes \mathcal{H}_{i_n}^0.$$

We shall describe the left action $\lambda_i: \mathbb{B}(\mathcal{H}_i) \rightarrow \mathbb{B}(\mathcal{H})$. It is convenient to introduce a subspace $\mathcal{H}(i) \subset \mathcal{H}$ which is the closed span of $\mathbb{C}\xi$ and those direct summands with $i \neq i_1$ in the above representation. Then, there is a canonical unitary operator

$$U_i: \mathcal{H} \rightarrow \mathcal{H}_i \otimes \mathcal{H}(i)$$

which identify $\mathcal{H}(i) \cong \mathbb{C}\xi_i \otimes \mathcal{H}(i)$ and $\mathcal{H}(i)^\perp \cong \mathcal{H}_i^0 \otimes \mathcal{H}(i)$. We define

$$\lambda_i: \mathbb{B}(\mathcal{H}_i) \ni a \mapsto U_i^*(a \otimes 1_{\mathcal{H}(i)})U_i \in \mathbb{B}(\mathcal{H}).$$

The reduced free-product C^* -algebra $(B, \tau) = (B_1, \tau_1) * (B_2, \tau_2)$ is the C^* -algebra B in $\mathbb{B}(\mathcal{H})$ generated by $\lambda_1(B_1)$ and $\lambda_2(B_2)$ with the distinguished trace $\tau(\cdot) = (\cdot, \xi)$ on B . We will omit λ_i when there are no confusions. The vector $\xi \in \mathcal{H}$ is a cyclic separating trace vector for B and the corresponding conjugation operator J is given by

$$J(\widehat{a}_1 \otimes \cdots \otimes \widehat{a}_n) = \widehat{a}_n^* \otimes \cdots \otimes \widehat{a}_1^* = (J_{i_n} \widehat{a}_n) \otimes \cdots \otimes (J_{i_1} \widehat{a}_1)$$

for $a_k \in B_{i_k}^0$ with $i_1 \neq i_2, \dots, i_{n-1} \neq i_n$. In particular, $J\mathcal{H}(i) \subset \mathcal{H}$ is the closed linear span of $\mathbb{C}\xi$ and $\widehat{a}_1 \otimes \cdots \otimes \widehat{a}_n$'s with $i_n \neq i$. Let $V_i: \mathcal{H} \rightarrow J\mathcal{H}(i) \otimes \mathcal{H}_i$ be the unitary operator which identifies $J\mathcal{H}(i) \cong J\mathcal{H}(i) \otimes \mathbb{C}\xi_i$ and $(J\mathcal{H}(i))^\perp \cong J\mathcal{H}(i) \otimes \mathcal{H}_i^0$. We note that V_i intertwines the right actions of \mathcal{M}_i . Moreover, the following is true.

Lemma 3.1. *We have*

$$\begin{aligned} \lambda_i(a) &= JV_i^*(1_{J\mathcal{H}(i)} \otimes J_i a J_i) V_i J \\ &= V_i^*(P_\xi \otimes a + \lambda_i(a)|_{J\mathcal{H}(i) \ominus \mathbb{C}\xi} \otimes 1_{\mathcal{H}_i}) V_i \\ &= V_j^*(\lambda_i(a)|_{J\mathcal{H}(j)} \otimes 1_{\mathcal{H}_j}) V_j \end{aligned}$$

for every $a \in \mathbb{B}(\mathcal{H}_i)$ and $j \neq i$, where P_ξ is the orthogonal projection onto $\mathbb{C}\xi$.

Proof. We first observe that the unitary operator

$$(J|_{J\mathcal{H}(i)} \otimes J_i) V_i J U_i^*: \mathcal{H}_i \otimes \mathcal{H}(i) \rightarrow \mathcal{H}(i) \otimes \mathcal{H}_i$$

is nothing but the flip of the tensor components \mathcal{H}_i and $\mathcal{H}(i)$. Hence, we have $JV_i^*(J \otimes J_i)(1 \otimes a)(J \otimes J_i) V_i J = U_i^*(a \otimes 1) U_i = \lambda_i(x)$ and the first equation follows.

We denote by P_i the orthogonal projection from \mathcal{H} onto $\mathbb{C}\xi \oplus \mathcal{H}_i^0$. We note that P_i commutes with $\lambda_i(a)$. Since $U_i V_i^*(\xi \otimes \zeta) = \zeta \otimes \xi$ for every $\zeta \in \mathcal{H}_i$, we have

$$V_i \lambda_i(a) P_i V_i^*(\xi \otimes \zeta) = V_i U_i^*(a\zeta \otimes \xi) = \xi \otimes a\zeta = (P_\xi \otimes a)(\xi \otimes \zeta)$$

for every $\zeta \in \mathcal{H}_i$. Hence, we have $\lambda_i(a) P_i = V_i^*(P_\xi \otimes a) V_i$. We observe that

$$(U_i \otimes 1_{\mathcal{H}_i}) V_i (1 - P_i) = (1_{\mathcal{H}_i} \otimes V_i) U_i (1 - P_i)$$

as a partial isometry from $(1 - P_i)\mathcal{H}$ onto $\mathcal{H}_i \otimes (\mathcal{H}(i) \cap J\mathcal{H}(i) \ominus \mathbb{C}\xi) \otimes \mathcal{H}_i$. It follows that

$$\begin{aligned} V_i^*(\lambda_i(a) \otimes 1) V_i (1 - P_i) &= V_i^*(U_i^* \otimes 1)(a \otimes 1 \otimes 1)(U_i \otimes 1) V_i (1 - P_i) \\ &= V_i^*(U_i^* \otimes 1)(a \otimes 1 \otimes 1)(1 \otimes V_i) U_i (1 - P_i) \\ &= V_i^*(U_i^* \otimes 1)(1 \otimes V_i) U_i (1 - P_i) \lambda_i(a) (1 - P_i) \\ &= \lambda_i(a) (1 - P_i). \end{aligned}$$

Since $Q_i := V_i (1 - P_i) V_i^*$ is the projection onto $(J\mathcal{H}(i) \ominus \mathbb{C}\xi) \otimes \mathcal{H}_i$, we have

$$\lambda_i(a) = \lambda_i(a) P_i + \lambda_i(a) (1 - P_i) = V_i^*((P_\xi \otimes a) + (\lambda_i(a) \otimes 1) Q_i) V_i$$

as we claimed. Finally, if $i \neq j$, then $(U_i \otimes 1_{\mathcal{H}_j}) V_j = (1_{\mathcal{H}_i} \otimes V_j) U_i$ and we have

$$V_j^*(\lambda_i(a)|_{J\mathcal{H}(j)} \otimes 1) V_j = U_i^*(1 \otimes V_j^*)(a \otimes 1 \otimes 1)(1 \otimes V_j) U_i = \lambda_i(a).$$

□

We denote $C_i = JB_i J$, $C = JBJ$ and $D_i = V_i^*(\mathbb{K}(J\mathcal{H}(i)) \otimes_{\min} \mathbb{B}(\mathcal{H}_i)) V_i$ for simplicity. The above lemma implies that $\lambda_i(a) D_j \subset D_j$ for any $i, j \in \{1, 2\}$ and $a \in \mathbb{B}(\mathcal{H}_i)$.

Proposition 3.2. *Let $(B, \tau) = (B_1, \tau_1) * (B_2, \tau_2)$, $\mathcal{H} = L^2 B$ and $C = JBJ$ be as above, and let $\Psi: \mathbb{B}(\mathcal{H}) \rightarrow \mathbb{B}(\mathcal{H})$ be a C -bimodule ucp map. If B_i are both exact and $D_i \subset \ker \Psi$ for both $i \in \{1, 2\}$, then the ucp map*

$$\tilde{\Psi}: B \otimes C \ni \sum_{k=1}^n a_k \otimes x_k \mapsto \Psi\left(\sum_{k=1}^n a_k x_k\right) \in \mathbb{B}(\mathcal{H})$$

is continuous w.r.t. the minimal tensor norm on $B \otimes C$.

Proof. For simplicity, we let $\tilde{B}_i = \lambda(\mathbb{B}(\mathcal{H}_i))$ and $\tilde{B} = C^*(\tilde{B}_1, \tilde{B}_2)$. We claim that $[\tilde{B}, C] \subset \ker \Psi$. Since Ψ is a C -bimodule map and the closed linear span of $\bigcup_{j=1,2} C[\tilde{B}, C_j]C$ contains $[\tilde{B}, C]$, it suffices to show $[\tilde{B}, C_j] \subset \ker \Psi$ for each $j \in \{1, 2\}$. But since $\tilde{B} D_k \tilde{B} \subset D_k$ by Lemma 3.1 and the closed linear span of $\bigcup_{i=1,2} \tilde{B}[\tilde{B}_i, C_j]\tilde{B}$ contains $[\tilde{B}, C_j]$, it suffices to show $[\tilde{B}_i, C_j] \subset \bigcup_{k=1,2} D_k$ for each $i, j \in \{1, 2\}$. Now, it is not hard to see from Lemma 3.1 that $[\tilde{B}_i, C_j] = \{0\}$ for $i \neq j$ and that

$$[\lambda_i(a), J\lambda_i(b)J] = V_i^*(\mathbb{C}P_\xi \otimes [a, J_i b J_i]) V_i \in D_i$$

for every $a \in \mathbb{B}(\mathcal{H}_i)$ and $b \in B_i$.

Since $[\tilde{B}, C] \subset \ker \Psi$ and Ψ is a C -bimodule map, we have $\Psi(\tilde{B}) \subset C'$. Lemma 2.4 implies that the inclusion map $\iota: B \hookrightarrow \tilde{B}$ is nuclear and so is the ucp map $\Psi_l = \Psi \circ \iota: B \rightarrow C'$. Therefore, the product ucp map

$$\tilde{\Psi} = \Psi_l \times \text{id}_C: B \otimes C \rightarrow \mathbb{B}(\mathcal{H})$$

is continuous w.r.t. the minimal tensor norm. \square

Theorem 3.3. *Let \mathcal{M}_1 and \mathcal{M}_2 be semiexact finite factors and let $\mathcal{M} = \mathcal{M}_1 * \mathcal{M}_2$ be their free-product. If $\mathcal{Q} \subset \mathcal{M}$ is an injective type II_1 subfactor whose relative commutant $\mathcal{Q}' \cap \mathcal{M}$ is a non-injective factor, then there exist $i \in \{1, 2\}$ and a unitary operator $u \in \mathcal{M}$ such that $u^* \mathcal{Q} u \subset \mathcal{M}_i$ in \mathcal{M} .*

Proof. We follow the notations used above. Let $B_i \subset \mathcal{M}_i$ be ultraweakly dense exact C^* -algebras, $(B, \tau) = (B_1, \tau_1) * (B_2, \tau_2)$ be their free-product and let $C = JBJ$. Then, B is ultraweakly dense in \mathcal{M} and is exact by Dykema's theorem [Dy2]. It follows Lemma 2.1 that the C -bimodule ucp map

$$\tilde{\Psi}_{\mathcal{Q}}: B \otimes C \ni \sum_{k=1}^n a_k \otimes x_k \mapsto \Psi_{\mathcal{Q}}\left(\sum_{k=1}^n a_k x_k\right) \in \mathbb{B}(L^2 \mathcal{M})$$

cannot be continuous w.r.t. the minimal tensor norm. But by Proposition 3.2, this implies that $D_i \not\subset \ker \Psi_{\mathcal{Q}}$ for some $i \in \{1, 2\}$. Now the discussion following Lemma 2.1 applies (for $\mathcal{N} = \mathcal{M}_i$) and yields a \mathcal{Q} - \mathcal{M}_i sub-bimodule \mathcal{K} in $L^2 \mathcal{M}$ with $\dim \mathcal{K}_{\mathcal{M}_i} < \infty$.

It follows Lemma 2.2 that there exist projections $e \in \text{Proj}(\mathcal{M}_i)$ and $q \in \text{Proj}(\mathcal{Q})$, a non-zero partial isometry $v \in \mathcal{M}$ and a homomorphism $\theta: q\mathcal{Q}q \rightarrow e\mathcal{M}_i e$ such that $vv^* \in (q\mathcal{Q}q)' \cap q\mathcal{M}q$, $v^*v \in \theta(q\mathcal{Q}q)' \cap e\mathcal{M}e$ and $xv = v\theta(x)$ for $x \in q\mathcal{Q}q$. Since $(q\mathcal{Q}q)' \cap q\mathcal{M}q = q(\mathcal{Q}' \cap \mathcal{M})q$, we have $vv^* = qq'$ for some $q' \in \text{Proj}(\mathcal{Q}' \cap \mathcal{M})$. By restricting θ and v if necessary, we may assume $\tau(q) = 1/m$ and $\tau(q') = 1/n$ for some $m, n \in \mathbb{N}$. Let $u_1, \dots, u_m \in \mathcal{Q}$ (resp. $u'_1, \dots, u'_n \in \mathcal{Q}' \cap \mathcal{M}$) be partial isometries such that $u_j^* u_j = q$ and $\sum_{j=1}^m u_j u_j^* = 1$ (resp. $(u'_k)^* u'_k = q'$ and $\sum_{k=1}^n u'_k (u'_k)^* = 1$). By Lemma 2.3, $v^*v \in e\mathcal{M}_i e$. We note that $\tau(v^*v) = \tau(qq') = (mn)^{-1}$. Let $w_{j,k}$ be partial isometries in \mathcal{M}_i such that $w_{j,k} w_{j,k}^* = v^*v$ and $\sum_{j=1}^m \sum_{k=1}^n w_{j,k}^* w_{j,k} = 1$. Then $u = \sum_{j,k} u_j u_k^* v w_{j,k}$ is the desired unitary operator. Indeed, $u^* x u = \sum_{j_1, j_2, k} w_{j_1, k}^* \theta(u_{j_1}^* x u_{j_2}) w_{j_2, k} \in \mathcal{M}_i$ for $x \in \mathcal{Q}$. \square

The Kurosh subgroup theorem states that if Λ is a subgroup in a free product $\Gamma_1 * \Gamma_2$, then Λ is freely generated by a free subgroup in $\Gamma_1 * \Gamma_2$ and/or conjugates of subgroups in Γ_1 and/or Γ_2 . In particular, if $\Lambda \leq \Gamma_1 * \Gamma_2$ is a freely-indecomposable non-infinite-cyclic subgroup, then Λ is conjugated to a subgroup in Γ_1 or Γ_2 . The following corollary is an analogue of this for type II_1 factors.

Corollary 3.4. *Let \mathcal{M}_1 and \mathcal{M}_2 be semiexact finite factors and let $\mathcal{M} = \mathcal{M}_1 * \mathcal{M}_2$ be their free-product. If $\mathcal{N} \subset \mathcal{M}$ is a non-prime non-injective subfactor whose relative commutant $\mathcal{N}' \cap \mathcal{M}$ is a factor, then there exist $i \in \{1, 2\}$ and a unitary operator $u \in \mathcal{M}$ such that $u^* \mathcal{N} u \subset \mathcal{M}_i$ in \mathcal{M} . In particular, \mathcal{M} is prime unless one of \mathcal{M}_i is trivial or both \mathcal{M}_i are isomorphic to $\mathbb{M}_2(\mathbb{C})$.*

Proof. Since \mathcal{N} is non-prime non-injective, there are II₁-factors \mathcal{N}_1 and \mathcal{N}_2 with \mathcal{N}_2 non-injective such that $\mathcal{N} = \mathcal{N}_1 \bar{\otimes} \mathcal{N}_2$. By Proposition 13 in [OP], there is an injective type II₁ subfactor $\mathcal{Q} \subset \mathcal{N}_1$ with $\mathcal{Q}' \cap \mathcal{M} = \mathcal{N}_1' \cap \mathcal{M}$. Since $\mathcal{N}_2 \subset \mathcal{N}_1' \cap \mathcal{M}$, the relative commutant $\mathcal{N}_1' \cap \mathcal{M}$ is non-injective and the center of $\mathcal{N}_1' \cap \mathcal{M}$ is contained in (the center of) $\mathcal{N}' \cap \mathcal{M}$. Hence the factoriality of $\mathcal{N}' \cap \mathcal{M}$ implies that $\mathcal{Q}' \cap \mathcal{M}$ is a non-injective factor. Now it follows from Theorem 3.3 that there exist $i \in \{1, 2\}$ and a unitary operator $u \in \mathcal{M}$ such that $u^* \mathcal{Q} u \subset \mathcal{M}_i$. If $v \in \mathcal{U}(u^* \mathcal{N}_2 u)$, then v commutes with $u^* \mathcal{Q} u$ and hence $v \in \mathcal{M}_i$ by Lemma 2.3. It follows that $u^* \mathcal{N}_2 u \subset \mathcal{M}_i$. The same argument applies to $u^* \mathcal{N}_1 u$ instead of $u^* \mathcal{Q} u$ and we have $u^* \mathcal{N}_1 u \subset \mathcal{M}_i$. Consequently, we have $u^* \mathcal{N} u \subset \mathcal{M}_i$. We note that \mathcal{M} is a non-injective factor unless one of \mathcal{M}_i is trivial or both \mathcal{M}_i are isomorphic to $\mathbb{M}_2(\mathbb{C})$ [Dy1]. Since \mathcal{M} is not unitarily conjugated into \mathcal{M}_i , it cannot be non-prime. \square

One of the consequences of the Kurosh subgroup theorem is the isomorphism theorem that if Γ_0 and Λ_0 are free groups and $\Gamma_1, \dots, \Gamma_n$ and $\Lambda_1, \dots, \Lambda_m$ are freely-indecomposable non-infinite-cyclic groups with $\Gamma = \ast_{i=0}^n \Gamma_i = \ast_{j=0}^m \Lambda_j$, then $\Gamma_0 = \Lambda_0$, $n = m$ and, modulo permutation of indices, Γ_i and Λ_i are conjugated in Γ for every $i \geq 1$. The following is an analogue of this for type II₁ factors.

Corollary 3.5. *Let $\mathcal{M}_0, \dots, \mathcal{M}_n$ and $\mathcal{N}_0, \dots, \mathcal{N}_m$ be semiexact finite factors such that \mathcal{M}_0 and \mathcal{N}_0 are semisolid (possibly one-dimensional) and that $\mathcal{M}_1, \dots, \mathcal{M}_n$ and $\mathcal{N}_1, \dots, \mathcal{N}_m$ are non-prime non-injective. If $\mathcal{M} = \ast_{i=0}^n \mathcal{M}_i = \ast_{j=0}^m \mathcal{N}_j$, then $n = m$ and, modulo permutation of indices, \mathcal{M}_i and \mathcal{N}_i are unitarily conjugated in \mathcal{M} for every $i \geq 1$.*

Proof. Without loss of generality, we assume $m \geq n$ (and we no longer need the assumption that \mathcal{N}_0 is semisolid). Let $\mathcal{M} = \ast_{i=0}^n \mathcal{M}_i = \ast_{j=0}^m \mathcal{N}_j$. By Corollary 3.4, there exist maps $\iota: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ and $j: \{1, \dots, n\} \rightarrow \{0, \dots, m\}$ and unitaries $u_1, \dots, u_m, v_1, \dots, v_n$ in \mathcal{M} such that $u_j^* \mathcal{N}_j u_j \subset \mathcal{M}_{\iota(j)}$ for every $j \geq 1$ and $v_i^* \mathcal{M}_i v_i \subset \mathcal{N}_{j(i)}$ for every $i \geq 1$. It follows that $v_{\iota(j)}^* u_j^* \mathcal{N}_j u_j v_{\iota(j)} \subset \mathcal{N}_{j(\iota(j))}$ for every $j \geq 1$. But by Lemma 2.3, this implies that $j(\iota(j)) = j$ and $u_j v_{\iota(j)} \in \mathcal{N}_j$. Therefore, the map ι is a bijection and the above inclusion maps are all isomorphisms. Hence, we have $u_j^* \mathcal{N}_j u_j = \mathcal{M}_{\iota(j)}$ for every $j \geq 1$. \square

4. CROSSED-PRODUCT

We recall the crossed product construction. Let A be a C^* -algebra with a Γ -action $\alpha: \Gamma \rightarrow \text{Aut}(A)$. A covariant representation of (Γ, A) is a pair (σ, π) of

representations of Γ and A respectively on a Hilbert space \mathcal{H} such that

$$\forall s \in \Gamma, \forall a \in A \quad \text{Ad}_{\sigma(s)}(\pi(a)) = \sigma(s)\pi(a)\sigma(s)^{-1} = \pi(\alpha_s(a)).$$

The full crossed product C^* -algebra $C^*(\Gamma, A)$ is the universal C^* -algebra generated by $\sigma(\Gamma)$ and $\pi(A)$ under the covariance condition. To define the reduced crossed product, we fix a faithful representation $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$. Define a new representation $\tilde{\pi}: A \rightarrow \mathbb{B}(\ell_2\Gamma \otimes \mathcal{H})$ by

$$\tilde{\pi}(a)(\delta_t \otimes \zeta) = \delta_t \otimes \pi(\alpha_t^{-1}(a))\zeta \quad \text{for } t \in \Gamma \text{ and } \zeta \in \mathcal{H}.$$

It is convenient to introduce orthogonal projections $e(t)$ of $\ell_2\Gamma$ onto $\mathbb{C}\delta_t$ so that $\tilde{\pi}(a) = \sum_{t \in \Gamma} e(t) \otimes \pi(\alpha_t^{-1}(a))$, where the sum converges strongly. It is easily verified that $(\lambda \otimes 1, \tilde{\pi})$ is a covariant representation of (Γ, A) . The reduced crossed product C^* -algebra $C_{\text{red}}^*(\Gamma, A)$ is the C^* -subalgebra in $\mathbb{B}(\ell_2\Gamma \otimes \mathcal{H})$ generated by $(\lambda \otimes 1)(\Gamma)$ and $\tilde{\pi}(A)$. We note that $C_{\text{red}}^*(\Gamma, A)$ does not depend on the choice of a faithful representation π . By definition, $C_{\text{red}}^*(\Gamma, A)$ is canonically isomorphic to a quotient of $C^*(\Gamma, A)$. If A is an exact C^* -algebra and Γ is an exact group, then $C_{\text{red}}^*(\Gamma, A)$ is exact also (cf. [KW]).

A compact Γ -space is a compact topological space X together with a continuous action of Γ on it. Recall that we say a compact Γ -space X is *amenable* (or, the Γ -action on X is amenable) if there exists a sequence of continuous $\mu^n: X \rightarrow \text{Prob}(\Gamma)$ such that

$$\lim_{n \rightarrow \infty} \sup_{x \in X} \|s \cdot \mu_x^n - \mu_{s \cdot x}^n\| = 0$$

for every $s \in \Gamma$, where $\text{Prob}(\Gamma) = \{\mu \in \ell_1\Gamma : \mu \geq 0, \|\mu\| = 1\}$ and $(s \cdot \mu)(t) = \mu(s^{-1}t)$ for $\mu \in \text{Prob}(\Gamma)$ and $s \in \Gamma$. We note that the Stone-Ćech compactification $\beta\Gamma$ with the left translation action of Γ is amenable iff Γ is exact [AD][GK][Oz1]. We assume that the reader is familiar with basic facts on amenability for group actions. We refer [AD][AR] for detail.

Let X be a compact Γ -space. A unital Γ - $C(X)$ - C^* -algebra is a unital C^* -algebra A such that

- (1) A contains $C(X)$ in its center,
- (2) there is a Γ -action $\alpha: \Gamma \rightarrow \text{Aut}(A)$,
- (3) $(\alpha_s(f))(x) = f(s^{-1} \cdot x)$ for every $s \in \Gamma$, $f \in C(X)$ and $x \in X$.

If the compact Γ -space X is amenable, then we have $C_{\text{red}}^*(\Gamma, A) = C^*(\Gamma, A)$ canonically for every Γ - $C(X)$ - C^* -algebra A .

We first prove a general result on amenability of a group action. Suppose that Γ acts on a set K . The Γ -action extends to a continuous action on the Stone-Ćech compactification βK of K and then restricts to a continuous action on the Stone-Ćech remainder $\partial^\beta K = \beta K \setminus K$. We are interested when the compact Γ -space $\partial^\beta K$ is amenable. It is well-known that the unitary representation $\sigma_K: \Gamma \rightarrow \mathbb{B}(\ell_2 K)$, given by $\sigma_K(s)\delta_x = \delta_{s \cdot x}$, is weakly contained in the left regular representation λ iff the isotropy subgroups are all amenable.

Proposition 4.1. *Let Γ be an exact group, K be a countable set on which Γ acts and $\partial^\beta K$ be the Stone-Ćech remainder of K . The following are equivalent.*

- (1) *The compact Γ -space $\partial^\beta K$ is amenable.*
- (2) *There exists a map $\mu: K \rightarrow \text{Prob}(\Gamma)$ such that $\lim_{x \rightarrow \infty} \|s \cdot \mu_x - \mu_{s \cdot x}\| = 0$ for every $s \in \Gamma$.*
- (3) *There exists a ucp map $\varphi: C_\lambda^* \Gamma \rightarrow \mathbb{B}(\ell_2 K)$ such that $\varphi(\lambda(s)) - \sigma_K(s) \in \mathbb{K}(\ell_2 K)$ for every $s \in \Gamma$.*

Proof. 1 \Rightarrow 3: We note that $C(\partial^\beta K) \cong \ell_\infty K / c_0 K \subset \mathbb{B}(\ell_2 K) / \mathbb{K}(\ell_2 K)$. If the compact Γ -space $\partial^\beta K$ is amenable, then the C^* -algebra $C_{\text{red}}^*(\Gamma, C(X))$ is nuclear and the natural homomorphism

$$C_\lambda^* \Gamma \subset C_{\text{red}}^*(\Gamma, C(X)) \cong C^*(\Gamma, C(X)) \rightarrow \mathbb{B}(\ell_2 K) / \mathbb{K}(\ell_2 K)$$

is continuous. Moreover, it has a ucp lifting φ by the Choi-Effros lifting theorem.

3 \Rightarrow 2: By a generalized Weyl-von Neumann theorem (Theorem II.5.3 in [Da]), there exists an isometry $V: \ell_2 K \rightarrow \ell_2 \Gamma$ such that $V^* \lambda(s) V - \varphi(\lambda(s)) \in \mathbb{K}(\ell_2 \Gamma)$ for every $s \in \Gamma$. For every $x \in K$, we set $\mu_x = |V \delta_x|^2 \in \text{Prob}(\Gamma)$. Then, we have

$$\begin{aligned} \|s \cdot \mu_x - \mu_{s \cdot x}\|_1 &\leq \|\lambda(s) V \delta_x + V \delta_{s \cdot x}\|_2 \|\lambda(s) V \delta_x - V \delta_{s \cdot x}\|_2 \\ &\leq 2(2 - 2\Re\langle V^* \lambda(s) V \delta_x, \delta_{s \cdot x} \rangle)^{1/2}. \end{aligned}$$

Since $V^* \lambda(s) V - \sigma_K(s) \in \mathbb{K}(\ell_2 \Gamma)$ for every $s \in \Gamma$, we are done.

2 \Rightarrow 1: Since the state space S of $\ell_\infty \Gamma$ is compact, the map $\mu: K \rightarrow \text{Prob}(\Gamma)$ extends to a continuous map $\tilde{\mu}$ from βK into S . By the condition 2, the map $\tilde{\mu}$ is Γ -equivariant on $\partial^\beta K$. Hence, the amenability of $\partial^\beta K$ follows from that of S . However, since the state space S is amenable iff the underlying space $\beta \Gamma$ is amenable, the amenability of S follows from the exactness of Γ . \square

Consider the $\Gamma \times \Gamma$ -action on Γ given by the left and right translations. Let \mathcal{S} be the class of countable discrete groups Γ such that the compact $\Gamma \times \Gamma$ -space $\partial^\beta \Gamma$ is amenable. The class \mathcal{S} contains all word hyperbolic groups (and more generally groups which are hyperbolic relative to a family of amenable subgroups) [HG][Oz3][Sk]. Moreover, every subgroups of a group in \mathcal{S} is again in \mathcal{S} and \mathcal{S} is closed under free-product (with finite amalgamation).

Although it is irrelevant to the rest of paper, we make the following observation. A group Γ is said to be inner-amenable if there exists a state m on $\ell_\infty \Gamma$ with $c_o \Gamma \subset \ker m$ such that $m(\sigma_s(f)) = m(f)$ for every $s \in \Gamma$ and $f \in \ell_\infty \Gamma$, where $\sigma_s(f)(t) = f(s^{-1}ts)$. In other words, Γ is inner-amenable if $\partial^\beta \Gamma$ carries a probability measure which is invariant under the conjugation action of Γ . (There is another definition of inner-amenability that requires only $m(\delta_e) = 0$ instead of $c_o \Gamma \subset \ker m$. But, they coincide if the group in consideration is ICC.) If $\Gamma \in \mathcal{S}$, then the conjugation action of Γ on $\partial^\beta \Gamma$ is amenable since it is the restriction of the $\Gamma \times \Gamma$ -action to its diagonal subgroup Γ . It follows that every inner-amenable group in \mathcal{S} is amenable.

This was first proved in [dH]. It is also not difficult to prove that every torsion-free non-amenable group in \mathcal{S} is ICC.

We look at the crossed product construction more carefully in the tracial setting. Let A be a C^* -algebra with a faithful trace τ and let $\alpha: \Gamma \rightarrow \text{Aut}(A, \tau)$ be a trace preserving action. If $\pi_\tau: A \rightarrow \mathbb{B}(L^2 A)$ is the GNS-representation of (A, τ) , then $C_{\text{red}}^*(\Gamma, A)$ is the C^* -subalgebra in $\mathbb{B}(\ell_2 \Gamma \otimes L^2 A)$ generated by $(\lambda \otimes 1)(\Gamma)$ and $\tilde{\pi}_\tau(A)$. The vector $\xi = \delta_e \otimes \widehat{1}_A$ is a cyclic separating trace vector for $C_{\text{red}}^*(\Gamma, A)$ with the canonical conjugation J on $\ell_2 \Gamma \otimes L^2 A$ given by $Jx\xi = x^*\xi$ for $x \in C_{\text{red}}^*(\Gamma, A)$. Let u be the representation of Γ on $L^2 A$ given by $u(s)\widehat{a} = \widehat{\alpha_s(a)}$ for $s \in \Gamma$ and $a \in A$. A simple calculation shows that

$$J(\lambda \otimes 1)(s)J = (\rho \otimes u)(s) \text{ for } s \in \Gamma \text{ and } J\tilde{\pi}_\tau(a)J = 1 \otimes \pi_\tau^c(a) \text{ for } a \in A,$$

where $\pi_\tau^c(a) = J_A \pi_\tau(a) J_A$. For simplicity, denote by $C^*(B, C)$ the C^* -subalgebra in $\mathbb{B}(\ell_2 \Gamma \otimes L^2 A)$ generated by $B = C_{\text{red}}^*(\Gamma, A)$ and $C = JBJ$.

Let $I_1 = \mathbb{K}(\ell_2 \Gamma) \otimes_{\min} \mathbb{B}(L^2 A)$. We note that I_1 is the hereditary C^* -subalgebra in $\mathbb{B}(\ell_2 \Gamma \otimes L^2 A)$. It is not hard to see that both B and C are in the multiplier of I_1 and hence $I = I_1 \cap C^*(B, C)$ is an ideal in $C^*(B, C)$.

Proposition 4.2. *Let $\alpha: \Gamma \rightarrow \text{Aut}(A, \tau)$ and $I \subset C^*(B, C)$ be as above. Suppose that $A'' \subset \mathbb{B}(L^2 A)$ is injective and $\Gamma \in \mathcal{S}$. Then, the homomorphism*

$$\nu: B \otimes C \ni \sum_{k=1}^n a_k \otimes x_k \mapsto \sum_{k=1}^n a_k x_k + I \in C^*(B, C)/I$$

is continuous w.r.t. the minimal tensor norm on $B \otimes C$.

Proof. Consider the C^* -algebra $D_1 = C^*(B, C, \ell_\infty \Gamma \otimes \mathbb{C}1)$ generated by $C^*(B, C)$ and $\ell_\infty \Gamma \otimes 1$. Since D_1 is in the multiplier of I_1 , there is a natural inclusion

$$C^*(B, C)/I \hookrightarrow (D_1 + I_1)/I_1 = E_1.$$

Let $D \subset D_1$ be the C^* -algebra generated by $\tilde{\pi}(A)$, $J\tilde{\pi}(A)J$ and $\ell_\infty \Gamma \otimes \mathbb{C}1$. Then, it is not hard to see that $E = (D + I_1)/I_1 \subset E_1$ is a $(\Gamma \times \Gamma)$ - $\partial\Gamma$ - C^* -algebra with the commuting Γ -actions $\text{Ad}_{\lambda \otimes 1}$ and $\text{Ad}_{\rho \otimes u}$. Since $\Gamma \in \mathcal{S}$, the canonical homomorphism from $C^*(\Gamma \times \Gamma, E)$ onto E_1 factors through $C_{\text{red}}^*(\Gamma \times \Gamma, E)$. Since $\tilde{\pi}(A)''$ is injective, the natural homomorphism $\tilde{\pi}(A) \otimes J\tilde{\pi}(A)J \rightarrow E$ is continuous w.r.t. the minimal tensor norm. Therefore, the homomorphism ν is continuous on $B \otimes_{\min} C \cong C_{\text{red}}^*(\Gamma \times \Gamma, \tilde{\pi}(A) \otimes_{\min} J\tilde{\pi}(A)J)$. \square

Let

$$K_1 = \{x \in \mathbb{B}(\ell_2 \Gamma \otimes L^2 A) : (\omega \otimes \text{id})(x^*x + xx^*) \in \mathbb{K}(L^2 A) \ \forall \omega \in \mathbb{B}(\ell_2 \Gamma)_*^+\}.$$

We note that K_1 is the hereditary C^* -subalgebra in $\mathbb{B}(\ell_2 \Gamma \otimes L^2 A)$ generated by $\ell_\infty(\Gamma, \mathbb{K}(L^2 A))$. It is not hard to see that both B and C are in the multiplier of K_1 and hence $K = K_1 \cap C^*(B, C)$ is an ideal in $C^*(B, C)$.

Proposition 4.3. *Let $\alpha: \Gamma \rightarrow \text{Aut}(A, \tau)$ and $K \subset C^*(B, C)$ be as above. Suppose that $A'' \subset \mathbb{B}(L^2 A)$ is injective and there exists a unital completely positive map $\theta: \mathbb{B}(L^2 A \otimes \ell_2 \Gamma) \rightarrow \mathbb{B}(L^2 A)$ such that the elements $\theta((u \otimes \rho)(s)) - u(s)$, $\theta(\pi_\tau(a) \otimes 1) - \pi_\tau(a)$ and $\theta(\pi_\tau^c(a) \otimes 1) - \pi_\tau^c(a)$ are all in $\mathbb{K}(L^2 A)$ for every $s \in \Gamma$ and $a \in A$. Then, the homomorphism*

$$\mu: B \otimes C \ni \sum_{k=1}^n a_k \otimes x_k \mapsto \sum_{k=1}^n a_k x_k + K \in C^*(B, C)/K$$

is continuous w.r.t. the minimal tensor norm on $B \otimes C$.

Proof. Since $A'' \subset \mathbb{B}(L^2 A)$ is injective, there is a canonical homomorphism

$$\Phi: (\mathbb{B}(\ell_2 \Gamma) \bar{\otimes} A'') \otimes_{\min} (A' \bar{\otimes} \mathbb{B}(\ell_2 \Gamma)) \rightarrow \mathbb{B}(\ell_2 \Gamma \otimes L^2 A \otimes \ell_2 \Gamma)$$

given by $\Phi(x \otimes 1 \otimes 1) = x \otimes 1$ for $x \in \mathbb{B}(\ell_2 \Gamma) \bar{\otimes} A''$ and $\Phi(1 \otimes 1 \otimes y) = 1 \otimes y$ for $y \in A' \bar{\otimes} \mathbb{B}(\ell_2 \Gamma)$. Let $U_0: \ell_2 \Gamma \otimes L^2 A \rightarrow L^2 A \otimes \ell_2 \Gamma$ be the unitary operator given by $U_0(\delta_t \otimes \xi) \mapsto u(t)\xi \otimes \delta_t$ so that the elements

$$\text{Ad}_{U_0}((\rho \otimes u)(s)) = (1 \otimes \rho)(s), \quad \text{Ad}_{U_0}(1 \otimes \pi_\tau^c(a)) = \sum_{t \in \Gamma} \pi_\tau^c(\alpha_t(a)) \otimes e(t)$$

are in $A' \bar{\otimes} \mathbb{B}(\ell_2 \Gamma)$. Let $U_1 = \sum_{t \in \Gamma} (\rho \otimes u)(t) \otimes e(t) \in B' \bar{\otimes} \mathbb{B}(\ell_2 \Gamma)$ be the unitary operator on $\ell_2 \Gamma \otimes L^2 A \otimes \ell_2 \Gamma$ so that

$$\text{Ad}_{U_1^*}((1 \otimes 1 \otimes \rho)(s)) = (\rho \otimes u \otimes \rho)(s), \quad \text{Ad}_{U_1^*(1 \otimes U_0)}(1 \otimes 1 \otimes \pi_\tau^c(a)) = 1 \otimes \pi_\tau^c(a) \otimes 1.$$

It follows that for the homomorphism

$$\tilde{\Phi} = \text{Ad}_{U_1^*} \Phi \text{Ad}_{1 \otimes 1 \otimes U_0}: B \otimes_{\min} C \rightarrow \mathbb{B}(\ell_2 \Gamma \otimes L^2 A \otimes \ell_2 \Gamma),$$

we have

$$\begin{aligned} \tilde{\Phi}((\lambda \otimes 1 \otimes 1 \otimes 1)(s)) &= \lambda(s) \otimes 1 \otimes 1, \\ \tilde{\Phi}(\tilde{\pi}_\tau(a) \otimes 1 \otimes 1) &= \tilde{\pi}_\tau(a) \otimes 1, \\ \tilde{\Phi}((1 \otimes 1 \otimes \rho \otimes u)(s)) &= (\rho \otimes u \otimes \rho)(s), \\ \tilde{\Phi}(1 \otimes 1 \otimes 1 \otimes \pi_\tau^c(a)) &= 1 \otimes \pi_\tau^c(a) \otimes 1. \end{aligned}$$

It follows that for the unital completely positive map

$$\tilde{\theta} = \text{id} \otimes \theta: \mathbb{B}(\ell_2 \Gamma \otimes L^2 A \otimes \ell_2 \Gamma) \rightarrow \mathbb{B}(\ell_2 \Gamma \otimes L^2 A),$$

the elements

$$\begin{aligned}
& \tilde{\theta}\tilde{\Phi}((\lambda \otimes 1 \otimes 1 \otimes 1)(s)) - \lambda(s) \otimes 1 = 0, \\
& \tilde{\theta}\tilde{\Phi}(\tilde{\pi}_\tau(a) \otimes 1 \otimes 1) - \tilde{\pi}_\tau(a) = \sum_{t \in \Gamma} e(t) \otimes (\theta(\pi_\tau(\alpha_t^{-1}(a)) \otimes 1) - \pi_\tau(\alpha_t^{-1}(a))), \\
& \tilde{\theta}\tilde{\Phi}((1 \otimes 1 \otimes \rho \otimes u)(s)) - (\rho \otimes u)(s) = \rho(s) \otimes (\theta((u \otimes \rho)(s)) - u(s)), \\
& \tilde{\theta}\tilde{\Phi}(1 \otimes 1 \otimes 1 \otimes \pi_\tau^c(a)) - 1 \otimes \pi_\tau^c(a) = 1 \otimes (\theta(\pi_\tau^c(a) \otimes 1) - \pi_\tau^c(a))
\end{aligned}$$

are all in K_1 for every $s \in \Gamma$ and $a \in A$. Since $(C^*(B, C) + K_1)/K_1 = C^*(B, C)/K$ canonically, the completely positive map $\tilde{\theta}\tilde{\Phi}: B \otimes_{\min} C \rightarrow C^*(B, C) + K_1$ passes to the homomorphism μ . \square

Let us recall the Bernoulli shift (or the wreath product) of Δ by Γ is the group $\Gamma \ltimes \Delta_\Gamma$, where $\Delta_\Gamma = \bigoplus_\Gamma \Delta$ is the direct sum of Δ 's indexed by Γ and Γ acts on Δ_Γ by left translation. We view an element $x \in \Delta_\Gamma$ as a function $x: \Gamma \rightarrow \Delta$ with finite support $\{t \in \Gamma : x(t) \neq e_\Delta\} =: \text{supp } x$.

Proposition 4.4. *Let Γ and Δ be groups with Δ amenable. Then, the Bernoulli shift action Γ on $A = C_\lambda^* \Delta_\Gamma$ satisfies the assumptions in Proposition 4.3.*

Proof. We note that $L^2 A$ is canonically isomorphic to $\ell_2 \Delta_\Gamma$. We fix a proper length function l_Γ on Γ , i.e., l_Γ is a non-negative function on Γ such that (i) $l_\Gamma(s) = 0$ iff $s = e$, (ii) $l_\Gamma(st) \leq l_\Gamma(s) + l_\Gamma(t)$ for $s, t \in \Gamma$, and (iii) the set $\{s \in \Gamma : l_\Gamma(s) \leq R\}$ is finite for every $R > 0$. Likewise l_Δ . For $y \in \Delta_\Gamma$ and $t \in \Gamma$, we set $w(y, t) = l_\Gamma(t) + l_\Delta(y(t))$ if $t \in \text{supp } y$ and $w(y, t) = 0$ otherwise. Further, set $w(y) = \sum_{t \in \Gamma} w(y, t)$ and $n(y) = |\text{supp } y|$. It follows that $n(y)/w(y) \rightarrow 0$ as $y \rightarrow \infty$. Define $\xi: \Delta_\Gamma \rightarrow \ell_2 \Gamma$ by $\xi_y(t) = (w(y, t)/w(y))^{1/2}$. Since $|w(y, t) - w(s(y), st)| \leq l_\Gamma(s)$ for $s \in \Gamma$ and $t \in \text{supp } y$, we have for every $s \in \Gamma$ that

$$\begin{aligned}
\|\lambda(s)\xi_y - \xi_{s(y)}\|_2^2 & \leq \|(\lambda(s)\xi_y)^2 - \xi_{s(y)}^2\|_1 \\
& = \sum_{t \in \Gamma} |w(y, t)/w(y) - w(s(y), st)/w(s(y))| \\
& = \left(\sum_{t \in \Gamma} |w(y, t) - w(s(y), st)|/w(y) \right) + |w(s(y)) - w(y)|/w(y) \\
& \leq 2l_\Gamma(s)n(y)/w(y) \rightarrow 0 \text{ as } y \rightarrow \infty.
\end{aligned}$$

Moreover for each $x \in \Delta_\Gamma$, we have $\|\xi_{xy} - \xi_y\|_2^2 \leq 2w(x)/w(y) \rightarrow 0$ and $\|\xi_{yx} - \xi_y\|_2^2 \leq 2w(x)/w(y) \rightarrow 0$ as $y \rightarrow \infty$.

Let $V: \ell_2 \Delta_\Gamma \rightarrow \ell_2 \Delta_\Gamma \otimes \ell_2 \Gamma$ be the isometry given by $V\delta_y = \delta_y \otimes \xi_y$. Then the unital completely positive map $\theta: \mathbb{B}(\ell_2 \Delta_\Gamma \otimes \ell_2 \Gamma) \rightarrow \mathbb{B}(\ell_2 \Delta_\Gamma)$, given by $\theta(z) = V^* z V$,

satisfies the assumption of Proposition 4.3 (with ρ replaced with λ). Indeed,

$$\begin{aligned} V^*(u \otimes \lambda)(s)V\delta_y &= \langle \lambda(s)\xi_y, \xi_{s(y)} \rangle \delta_{s(y)} \\ V^*(\lambda(x) \otimes 1)V\delta_y &= \langle \xi_y, \xi_{xy} \rangle \delta_{xy} \\ V^*(\rho(x) \otimes 1)V\delta_y &= \langle \xi_y, \xi_{yx^{-1}} \rangle \delta_{yx^{-1}} \end{aligned}$$

for every $s \in \Gamma$ and $x \in \Delta_\Gamma$. \square

The group Γ acts on Δ_Γ by Bernoulli shift and $\Delta_\Gamma \times \Delta_\Gamma$ acts on Δ_Γ by left and right translations. These actions induce an action of $\Lambda := \Gamma \ltimes (\Delta_\Gamma \times \Delta_\Gamma)$ on the Stone-Ćech remainder $\partial^\beta \Delta_\Gamma$. We observe from the above proof that this action is amenable provided that Γ is exact. Indeed, the map $\xi^2: \Delta_\Gamma \rightarrow \ell_1 \Gamma$ gives rise to a continuous map from the Stone-Ćech compactification $\beta \Delta_\Gamma$ into the state space of $\ell_\infty \Gamma$ whose restriction to $\partial^\beta \Delta_\Gamma$ is Λ -equivariant (where $\Delta_\Gamma \times \Delta_\Gamma$ acts trivially on $\ell_\infty \Gamma$). Since Δ_Γ is amenable there exists a Λ -equivariant conditional expectation from $\ell_\infty \Lambda$ onto $\ell_\infty \Gamma$. Composing these maps, we obtain a Λ -equivariant continuous map from $\partial^\beta \Delta_\Gamma$ into the state space of $\ell_\infty \Lambda$. Therefore, the amenability of $\partial^\beta \Delta_\Gamma$ follows from that of the state space of $\ell_\infty \Lambda$, which is amenable when Γ (and hence Λ) is exact.

Corollary 4.5. *If $\Gamma \in \mathcal{S}$ and Δ is amenable, then the Bernoulli shift $\Gamma \ltimes \Delta_\Gamma$ is in \mathcal{S} .*

Proof. By Propositions 4.2, 4.3 and 4.4 (and their proof), the homomorphism

$$B \otimes C \ni \sum_{k=1}^n a_k \otimes x_k \mapsto \sum_{k=1}^n a_k x_k + (I \cap K) \in C^*(B, C)/(I \cap K)$$

is continuous w.r.t. the minimal norm and has a ucp lifting. Since $I_1 \cap K_1 = \mathbb{K}(\ell_2 \Gamma \otimes \ell_2 \Delta_\Gamma)$, the claim follows from Proposition 4.1. \square

If $A = \mathcal{A}$ is a von Neumann algebra, then the crossed product von Neumann algebra is $\Gamma \ltimes \mathcal{A} = C_{\text{red}}^*(\Gamma, \mathcal{A})'' \subset \mathbb{B}(\ell_2 \Gamma \otimes L^2 \mathcal{A})$. The following are the main results of this section.

Theorem 4.6. *Let (\mathcal{A}, τ) be a commutative von Neumann algebra with a faithful trace τ and a trace preserving action $\alpha: \Gamma \rightarrow \text{Aut}(\mathcal{A}, \tau)$ and let $\mathcal{M} = \Gamma \ltimes \mathcal{A}$ be its crossed product (which may not be a factor). If $\Gamma \in \mathcal{S}$, then \mathcal{M} is semisolid. In particular, any non-injective subfactor of \mathcal{M} is prime.*

Proof. We follow the notations used above. In particular, $B = C_{\text{red}}^*(\Gamma, A)$ and $C = JBJ$. Let $\mathcal{Q} \subset \mathcal{M}$ be a type II₁ von Neumann subalgebra. Passing to a subalgebra if necessary, we may assume \mathcal{Q} is injective. For a proof by contradiction, suppose that $\mathcal{Q}' \cap \mathcal{M}$ is not injective. It follows from Lemma 2.1 that the ucp map

$$\tilde{\Psi}_{\mathcal{Q}}: B \otimes C \ni \sum_{k=1}^n a_k \otimes x_k \mapsto \Psi_{\mathcal{Q}}\left(\sum_{k=1}^n a_k x_k\right) \in \mathbb{B}(L^2 \mathcal{M})$$

cannot be continuous w.r.t. the minimal tensor norm. But, by Proposition 4.2, this implies that $\mathbb{K}(\ell_2\Gamma) \otimes \mathbb{B}(L^2\mathcal{A}) \not\subset \Psi_{\mathcal{Q}}$. Now the discussion following Lemma 2.1 applies (for $\mathcal{N} = \mathcal{A}$) and yields a non-zero \mathcal{Q} - \mathcal{A} bimodule \mathcal{H} with $\dim \mathcal{H}_{\mathcal{A}} < \infty$. This is absurd since \mathcal{A} is commutative while \mathcal{Q} is of type II_1 . \square

Theorem 4.7. *Let Γ be an exact group and \mathcal{R} be the hyperfinite type II_1 factor. Consider the Bernoulli product $\mathcal{M} = \Gamma \ltimes \bar{\otimes}_{\Gamma} \mathcal{R}$. Then, for any diffuse von Neumann subalgebra $\mathcal{Q} \subset \bar{\otimes}_{\Gamma} \mathcal{R}$, the relative commutant $\mathcal{Q}' \cap \mathcal{M}$ is injective.*

Proof. Let Δ be an amenable ICC group so that $L\Delta = \mathcal{R}$ and let $A = C_{\lambda}^*\Delta_{\Gamma}$. Let a diffuse von Neumann subalgebra $\mathcal{Q} \subset \bar{\otimes}_{\Gamma} \mathcal{R} = A''$ be given. We will follow the proof of Theorem 6 in [Oz2]. Passing to a von Neumann subalgebra if necessary, we may assume that \mathcal{Q} is generated by a single unitary operator $v \in A''$ with $\lim_k v^k = 0$ ultraweakly. Fix a non-trivial ultrafilter ω and consider the proper conditional expectation $\Psi_{\mathcal{Q}}$ from $\mathbb{B}(\ell_2\Gamma \otimes L^2A)$ onto $\tilde{\pi}_{\tau}(\mathcal{Q})'$ given by

$$\Psi_{\mathcal{Q}}(x) = \text{weak}^* - \lim_{n \rightarrow \omega} n^{-1} \sum_{k=1}^n \text{Ad}_{\tilde{\pi}_{\tau}(v^k)}(x).$$

Then, for any $x = \sum_{t \in \Gamma} e(t) \otimes x(t) \in \ell_{\infty}(\Gamma, \mathbb{K}(L^2A))$, we have

$$\Psi_{\mathcal{Q}}(x) = \text{weak}^* - \lim_{n \rightarrow \omega} \sum_{t \in \Gamma} e(t) \otimes u(t)^* \left(n^{-1} \sum_{k=1}^n \text{Ad}_{\pi_{\tau}(v^k)}(u(t)x(t)u(t)^*) \right) u(t) = 0$$

since each $u(t)x(t)u(t)^*$ is a compact operator. In particular, $\Psi_{\mathcal{Q}}(K) = 0$. It follows from Lemma 2.1 and Proposition 4.3 that the relative commutant $\mathcal{Q}' \cap \mathcal{M}$ is injective. \square

Remark 4.8. A similar proof applies to $\mathcal{M} = \Gamma \ltimes \mathcal{R}$, where $\Gamma \in \mathcal{S}$ and \mathcal{R} is the hyperfinite type II_1 factor. (Even though the C^* -algebra $C_{\text{red}}^*(\Gamma, \mathcal{R})$ is not exact, Lemma 2.1 is applicable since the pair $C_{\text{red}}^*(\Gamma, \mathcal{R}) \subset \mathcal{M}$ satisfies local reflexivity.) It follows that if $\mathcal{Q} \subset \mathcal{M}$ is a non-injective subfactor, then the relative commutant $\mathcal{Q}' \cap \mathcal{M}$ is injective. In particular, any non-McDuff subfactor of \mathcal{M} is prime. This result applies to the factors constructed in [NPS].

Consider an essentially-free measure-preserving ergodic action of a non-amenable hyperbolic group Γ on the standard probability space $[0, 1]$ and let

$$\mathcal{A} = L^{\infty}[0, 1] \subset \Gamma \ltimes L^{\infty}[0, 1] = \mathcal{M}.$$

Then, Adams' theorem [Ad] says $(\mathcal{A} \subset \mathcal{M}) \neq (\mathcal{A}_1 \subset \mathcal{M}_1) \bar{\otimes} (\mathcal{A}_2 \subset \mathcal{M}_2)$ for any type II_1 factors \mathcal{M}_i with Cartan subalgebras $\mathcal{A}_i \subset \mathcal{M}_i$. Combined with Popa's theorem [Po2], this implies that $\mathcal{M} \neq \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ for any HT type II_1 factors \mathcal{M}_i . Theorem 4.6 generalizes these to that $\mathcal{M} \neq \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ for any type II_1 factors \mathcal{M}_i .

By considering an ergodic but not strongly-ergodic action of $\Gamma \in \mathcal{S}$, we obtain a semisolid (and hence prime) type II_1 factor with the property (Γ) .

Voiculescu [Vo1] proved that the free group factors $L\mathbb{F}_r$ do not have regular diffuse injective subalgebras. In particular, $L\mathbb{F}_r$ do not have Cartan subalgebras. The Bernoulli shift $\mathbb{F}_r \ltimes (\bigotimes_{\mathbb{F}_r} \mathcal{R})$ of \mathcal{R} by \mathbb{F}_r is an example of a solid factor with a Cartan subalgebra.

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